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Energy correlations between a damaged macroscopic continuum and its sub-scale

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Abstract

An energy correlation hypothesis between a damaged macroscopic continuum and its sub-scale virgin, or matrix, material is established in the present paper. Two energy equivalence principles are proposed based on the geometrical definitions of the damage parameter and thermodynamic principles, which give relationships between macroscopic and effective definitions of stress and strain. These relationships are used to obtain yield condition and plastic flow rule for the damaged material when the mechanical properties of undamaged virgin, or matrix, material are given. It is shown that a link between the void growth model (VGM) and the continuum damage mechanics (CDM) exists when a more general definition of effective stress is used. Contradictory damage parameters based on different damage measuring techniques and equivalence hypotheses in CDM are clarified. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The development of material damage has been recognized as an important factor during a ductile failure process. Various theories and models have been proposed to study the influence of damage on structural response and failure. Two widely accepted methods are continuum damage mechanics (CDM) and the void growth model (VGM).

CDM was proposed by Kachanov (1958) for the creep failure of metals under uniaxial loads. This concept was taken up again in the seventies and extended to ductile and fatigue material

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failures within the framework of the thermodynamics of irreversible processes (Lemaitre and Chaboche, 1990).

A CDM model is normally based on two basic concepts, i.e., effective stress and an equivalence hypothesis. Various equivalence hypotheses have been proposed in order to transform the deformation state in macroscopic scale into its equivalent sub-scale where the virgin material is assumed to be damage-free (Lemaitre and Chaboche, 1990). The strain equivalence hypothesis proposed by Lemaitre (1971) and the elastic strain energy equivalence hypothesis introduced by Sidoroff (1981) and Cordebois and Sidoroff (1983) have been widely used in developing a CDM model. For example, Simo and Ju (1987) used the strain equivalence hypothesis in a continuum damage model and Ju (1989) established a coupled elastoplastic damage theory based on the strain equivalence hypothesis. Kattan and Voyiadjis (1990) and Voyiadjis and Kattan (1990) employed the elastic strain energy equivalence hypothesis to formulate a coupled theory of elasticity and continuum damage mechanics. Chow and Wang (1987a, b) used the elastic strain energy equivalence hypothesis to establish an anisotropic theory of continuum damage mechanics. In this case, the elastic strain energy equivalence hypothesis produces asymmetry of the stiffness matrix when anisotropic damage is considered (Chow and Wang, 1987a). Some authors have pointed out that the energy type equivalence hypothesis may be of more physical significance from the viewpoint of energy conservation (Zhu and Cescotto, 1992). The existing hypothesis of elastic strain energy equivalence (Sidoroff, 1981) postulated that the complementary elastic strain energy for a damage material is the same in form as that of an undamaged material, except that the stress is replaced by the effective stress in the energy formulation. Hansen and Schreyer (1994) discussed the relationship between strain equivalence and elastic strain energy equivalence hypotheses. It was shown that the choice between these two different hypotheses has a marked difference on the behaviour of the damage model. Thus, it is necessary to examine physical significance of an equivalence hypothesis.

The void growth model (VGM) treats the damaged material as a composite of void and matrix material. Matrix material can be treated as a damage-free material, which has the same physical meaning as the virgin material in CDM. Thus, both CDM and VGM employ the same viewpoint that the macroscopic damage or void developments are due to the plastic deformation of damage-free virgin material, or matrix material, in mesoscale (damage or void-size scale). In the following discussion, the term, ‘matrix material’, has the same meaning as ‘virgin material’, and will be used in most places in both CDM and VGM models.

Based on the understanding of a single void development on mesoscale for different basic void geometries, a proper average technique over the macroscopic scale may lead to some macroscopic mechanical properties of the damaged materials (Rice and Tracey, 1967; McClintock, 1968; Thomason, 1990). One successful model of this type is Gurson's (1977) model, which leads to a macroscopic yield condition including material damage effects. This model was modified by Tvergaard (1981) and Tvergaard and Needleman (1984) to consider the interactions between voids, which have been summarized by Tvergaard (1990), and have been used widely in predicting material ductile failures.

VGM faces the same problem as in CDM when transforming the deformation state in macroscopic scale into its equivalent sub-scale on matrix material level. In VGM, an assumption of plastic strain energy conservation in an average sense is employed to accomplish this transformation. VGM is valid within the plastic deformation range, while CDM is applicable in both the elastic and inelastic ranges. It appears that CDM is more likely to be a phenomenological model based on thermodynamics and VGM is an average technique for a plastic boundary value problem in the mesoscale. No matter how different the basic foundations are between them, they have been used to solve the same class of problems when material damage effects on the structural response and failure are significant. The mechanical properties of the damage-free matrix material were used in both models to obtain the macroscopic material constitutive equations in a damaged state. Unfortunately, little attention has been paid to the relationship between these two theories. Thus, it is worth to explore the possible relationships between them, which may lead to further development of the existing material damage theories.

A hypothesis of energy correlation between a macroscopic damaged continuum and its equivalent sub-scale in matrix material level is introduced in the present paper. Some basic concepts are clarified in section 2. Thermodynamic foundations are presented in section 3, based on which energy correlations of different mechanical processes between two material scales are obtained to establish relationships between macroscopic and effective definitions of stress and strain. These results are then used in section 4 to examine the existing equivalence hypotheses and reveal the relationship between CDM and VGM.

2. Basic concepts

For simplicity, the following discussion concentrates on isotropic plasticity and damage, although anisotropy is an intrinsic feature of a damaged material (Chow and Wang, 1987a, b).

2.1. Damage definitions

There are various ways to define material damage (Woo and Li, 1993). A widely accepted definition in the macroscopic scale is the geometrical description of material damage. In CDM, the area density of damage, proposed by Kachanov (1958), is defined by

$$D = \frac{A - A_s}{A} \tag{1}$$

where A is the total cross-section area of a surface within the unit cell in one of the three perpendicular directions, and A_s is the solid matrix area within A. In VGM, the volume density of damage is defined by (Tvergaard, 1981, 1990; Tvergaard and Needleman, 1984)

$$f = \frac{V - V_s}{V} \tag{2}$$

where V is the total volume of the unit cell and V_s is the solid matrix volume in the cell. Similarly, a concept of line density of damage, l, may be introduced as

$$l = \frac{L - L_s}{L} \tag{3}$$

where L is the characteristic length of the unit cell in one of the three perpendicular directions, and L_s is the characteristic length of solid matrix in L. It should be noted that these damage densities are defined at a ‘macroscopic point’ that is large enough to be statistically representative of the properties of the damaged material. D, f and l are understood as statistical averages among all possible values in the selected cell.

From the regular arrangements of voids in a cell, it is not difficult to obtain the following relationship (see Appendix)

$$f = \alpha D^{\frac{3}{2}} \tag{4}$$

where $\alpha = 0.752$ and $\alpha = 1$ for spherical and cubic voids, respectively. Because D and f are statistical averages among the unit material cell, Eq. (4) does not depend on a particular arrangement of the voids.

2.2. Macroscopic stress and effective stress

Macroscopic, or homogenized (Ju, 1989), stress and strain are defined as the conventional stress and strain for a unit material cell of a 'macroscopic point'. In metal deformation, Euler stress (true stress) and natural strain (true strain) are used in the constitutive equation.

Effective stress is defined as the force divided by the 'effective area' of the unit cell of a damaged 'macroscopic point'. The effective area of the unit cell is A_s or $(1-D)A$. Therefore, the relationship between macroscopic stress (σ_{ij}) and the corresponding effective stress ($\bar{\sigma}_{ij}$) in a damaged material is

$$\bar{\sigma}_{ij} = \frac{\sigma_{ij}}{1-D} \tag{5}$$

The corresponding strain of an effective stress is called effective strain in CDM. Eq. (5) is a mathematical definition of the effective stress although it may be understood as the average stress acting on an effective area of the material. In order to give it a general physical meaning, it is necessary to use the corresponding damage-free matrix material in mesoscale to represent the 'effective' concept in Eq. (5) for a macroscopically damaged material and a proper correlating hypothesis between two material scale levels is required.

2.3. Virgin state and correlating hypothesis

For a damaged material, it is assumed that there exists a corresponding virgin state which is the same material when all damages are removed (Lemaitre and Chaboche, 1990). The effective stress is the stress applied to the virgin material. The existence of an effective stress in the virgin state and an equivalence hypothesis are required as the theoretical foundation of CDM. In VGM, similar concepts are also used when developing a constitutive equation where the stress and strain of undamaged matrix material are introduced. Thus, as noted before, the term 'matrix material' in VGM has the same meaning as the term 'virgin material' in CDM.

In order to correlate the pre-assumed virgin state and the actual damaged state, several equivalence hypotheses have been proposed, which will be discussed further in section 4. In the present paper, Freudenthal's (1950, pp. 20) proposal is employed to use an energy concept to correlate phenomena between the virgin and damaged states, "Correlation of behaviour on the different levels is possible only in terms of a concept which on all levels has the same meaning in both Newtonian and statistical mechanics, the same dimension, and the same tensorial rank. This concept is energy. Being a scalar, that is, a tensor of rank zero, it is an algebraically additive quantity and has the same meaning on all levels of group of formation." A similar idea was also emphasized by Gordon (1976) "You may ignore something in different scales, but you cannot ignore energy which is a common concept in every scale."

It has been shown that the energy concept also plays an important role in material failure criteria (Chaouadi et al., 1994; Shen and Jones, 1992), which will be studied in a separate paper (Li, 1999a). The following hypothesis based on the energy concept is proposed in the present paper to obtain energy equivalence principles.

Energy Correlating Hypothesis: Each type of energy process in a damaged material is the same as the corresponding one in its equivalent virgin state.

3. Thermodynamic foundation and energy equivalent principles

3.1. Thermodynamic foundation

It is assumed that the damaged material obeys the first and second laws of thermodynamics and the additive assumption of elastic and plastic strains in macroscopic scale, and thus, the following equations may be obtained (Lemaitre and Chaboche, 1990)

$$\sigma_{ij} = \rho \frac{\partial \psi}{\partial \epsilon_{ij}^{\mathrm{e}}} \tag{6a}$$

$$s = -\frac{\partial \psi}{\partial T} \tag{6b}$$

$$\mathrm{d}\phi = \sigma_{ij}\, \mathrm{d}\epsilon_{ij}^{\mathrm{p}} + B\, \mathrm{d}D - A_{\mathrm{k}}\, \mathrm{d}V_{\mathrm{k}} - \frac{\nabla T}{T}\, \mathrm{d}q \geq 0 \tag{6c}$$

in which ρ is the damaged material density, T is temperature, s is the specific entropy density of the damaged material, q is the heat flux vector, D and V_{k} are internal variables to represent material damage and other internal structural changes, ψ is the free specific energy density of the damaged material defined by

$$\psi = e - Ts = \psi(\epsilon_{ij}^{\mathrm{e}}, T, D, V_{\mathrm{k}}) \tag{7}$$

where e is the specific internal energy density, and B and A_{k} are defined by

$$B = -\rho \frac{\partial \psi}{\partial D} \quad \text{and} \quad A_{\mathrm{k}} = \rho \frac{\partial \psi}{\partial V_{\mathrm{k}}} \tag{8}$$

For an isothermal elastic process, the increment of the free energy density of the damaged material is

$$\mathrm{d}(\rho\psi) = \frac{\partial(\rho\psi)}{\partial \epsilon_{ij}^{\mathrm{e}}}\, \mathrm{d}\epsilon_{ij}^{\mathrm{e}} = \sigma_{ij}\, \mathrm{d}\epsilon_{ij}^{\mathrm{e}} = \mathrm{d}W^{\mathrm{e}} \tag{9}$$

according to Eq. (6a), where W^{e} is the elastic strain energy density of the damaged material.

It is reasonable to assume that the mechanical dissipation due to any internal structure changes and thermal dissipation due to conduction of heat are independent (pp. 64 in Lemaitre and Chaboche, 1990). Eq. (6c) may be expressed equally by

$$\mathrm{d}\phi_1 = \sigma_{ij}\, \mathrm{d}\epsilon_{ij}^{\mathrm{p}} + B\, \mathrm{d}D - A_{\mathrm{k}}\, \mathrm{d}V_{\mathrm{k}} \geq 0 \tag{10a}$$

$$\mathrm{d}\phi_2 = -\frac{\nabla T}{T}\, \mathrm{d}q \geq 0, \tag{10b}$$

in which the increment of mechanical dissipation consists of three parts for an elastoplastic damage process, i.e. plastic dissipation, damage dissipation and cold work dissipation. It has been shown that the cold work dissipation represents only 5–10% of the plastic dissipation, which is neglected in the present analysis. Therefore, the increment of the mechanical energy dissipation density is

$$\mathrm{d}\phi_1 = \sigma_{ij}\,\mathrm{d}\epsilon^{\mathrm{p}}_{ij} + B\,\mathrm{d}D \tag{11}$$

for the damaged material.

On the other hand, the increments of the elastic strain energy and the mechanical energy dissipation for the corresponding matrix material within the unit cell of a damaged material are

$$\mathrm{d}\bar{W}^{\mathrm{e}} = \bar{\sigma}_{ij}\,\mathrm{d}\bar{\epsilon}^{\mathrm{e}}_{ij}(1-f) \tag{12a}$$

$$\mathrm{d}\bar{\phi}_1 = \bar{\sigma}_{ij}\,\mathrm{d}\bar{\epsilon}^{\mathrm{p}}_{ij}(1-f), \tag{12b}$$

in which, factor $(1-f)$ represents a reduction of material volume from damaged material to its corresponding virgin, or matrix, material.

3.2. Energy equivalence principles

There are two different energy processes in an elastoplastic damaged material, i.e. an elastic process and a dissipative process. According to Lee (1981), elastic and plastic deformations can be decomposited in an elastoplastic deformation. This method has been used for a general elastoplastic-damage dissipation problem (Li, 1999b). Thus, the energy types associated with elastic and inelastic (dissipative) processes in a general elastoplastic-damage dissipation problem belong to different energy types. According to the energy correlating principle proposed in section 2.3, the following two energy equivalent principles are derived from the thermodynamic results outlined in section 3.1.

Energy Equivalent Principle I: The increment of the elastic strain energy of a unit cell of the damaged material is correlated with the increment of the elastic strain energy in its corresponding matrix material, i.e.

$$\sigma_{ij}\,\mathrm{d}\epsilon^{\mathrm{e}}_{ij} = \bar{\sigma}_{ij}\,\mathrm{d}\bar{\epsilon}^{\mathrm{e}}_{ij}(1-f); \tag{13}$$

Energy Equivalent Principle II: The increment of mechanical dissipative energy in a damaged material during a dissipative process is correlated with the increment of mechanical dissipative energy in its corresponding matrix material, i.e.

$$\sigma_{ij}\,\mathrm{d}\epsilon^{\mathrm{p}}_{ij} + B\,\mathrm{d}D = \bar{\sigma}_{ij}\,\mathrm{d}\bar{\epsilon}^{\mathrm{p}}_{ij}(1-f). \tag{14}$$

It should be noted that Energy Equivalent Principle I is expressed in an incremental form and includes the material volume reduction due to material damage, which are not taken into account in the existing hypothesis of elastic strain energy equivalence. An incremental expression is equivalent to an integral expression when elastic deformation has no influence on damage evolution, which is used in the following analysis in section 3.3. Energy Equivalent Principle II is slightly different from the existing plastic strain energy equation in VGM (Tvergaard, 1981, 1990; Tvergaard and Needleman, 1984) because the damage term, $B\,\mathrm{d}D$, is included in the currently proposed Energy Equivalent Principle II for a complete dissipative energy correlation between damaged material and its corresponding matrix material. Further discussion will be given in section 3.3.

3.3. Results

The von-Mises equivalent stress and equivalent elastic strain as well as the hydrostatic stress and elastic volumetric strain of a damaged material are defined by

$$\sigma_{\mathrm{e}} = \sqrt{\frac{3}{2}(\sigma_{ij} - \sigma_{\mathrm{H}}\delta_{ij})(\sigma_{ij} - \sigma_{\mathrm{H}}\delta_{ij})}, \tag{15a}$$

$$\epsilon_{\mathrm{e}} = \sqrt{\frac{2}{3}(\epsilon^{\mathrm{e}}_{ij} - \epsilon^{\mathrm{e}}_{\mathrm{H}}\delta_{ij})(\epsilon^{\mathrm{e}}_{ij} - \epsilon^{\mathrm{e}}_{\mathrm{H}}\delta_{ij})}, \tag{15b}$$

$$\sigma_{\mathrm{H}} = \frac{1}{3}\sigma_{\mathrm{kl}}\delta_{\mathrm{kl}}, \tag{15c}$$

$$\epsilon^{\mathrm{e}}_{\mathrm{H}} = \frac{1}{3}\epsilon_{\mathrm{kl}}\delta_{\mathrm{kl}}, \tag{15d}$$

respectively, which satisfy the following relationships

$$\sigma_{\mathrm{e}} = \frac{3}{2}\,\frac{E}{1+\nu}\epsilon_{\mathrm{e}} \tag{16a}$$

$$\sigma_{\mathrm{H}} = \frac{E}{1-2\nu}\epsilon^{\mathrm{e}}_{\mathrm{H}}. \tag{16b}$$

The elastic strain energy density of a damaged material is

$$W^{\mathrm{e}} = W^{\mathrm{e}}_{\mathrm{d}} + W^{\mathrm{e}}_{\mathrm{v}} = \frac{\sigma^2_{\mathrm{e}}}{2E}\left[\frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_{\mathrm{H}}}{\sigma_{\mathrm{e}}}\right)^2\right], \tag{17}$$

where

$$W^{\mathrm{e}}_{\mathrm{d}} = \frac{(1+\nu)}{3E}\sigma^2_{\mathrm{e}} \tag{18a}$$

$$W^{\mathrm{e}}_{\mathrm{v}} = \frac{3(1-2\nu)}{2E}\sigma^2_{\mathrm{H}} \tag{18b}$$

are the elastic distortional and dilatational strain energy densities, respectively. Similar equations can be derived for the corresponding matrix material when any physical quantity [ ] of the damaged material in Eqs. (15)–(18) is substituted by its corresponding undamaged virgin state $[\bar{\ }]$.

For a damaged material, there is no change in the plastic and damage internal variables during any elastic processes. Suppose the elastic parameters of the damaged material depend on the damage parameter D. Now, let us apply an elastic distortional deformation and an elastic dilatational deformation on the damaged material, individually. According to the Energy Equivalent Principle I which is given in section 3.2, the following integral relationships between the damaged material and its corresponding virgin material are obtained when the elastic deformation process is assumed to have no influence on damage evolution

$$W^{\mathrm{e}}_{\mathrm{d}} = \bar{W}^{\mathrm{e}}_{\mathrm{d}}(1-f) \tag{19a}$$

$$W^{\mathrm{e}}_{\mathrm{v}} = \bar{W}^{\mathrm{e}}_{\mathrm{v}}(1-f), \tag{19b}$$

or,

$$\frac{1+\nu}{E}\sigma_{\mathrm{e}}^{2}=\frac{1+\bar{\nu}}{\bar{E}}\bar{\sigma}_{\mathrm{e}}^{2}(1-f) \tag{20a}$$

$$\frac{1-2\nu}{E}\sigma_{\mathrm{H}}^{2}=\frac{1-2\bar{\nu}}{\bar{E}}\bar{\sigma}_{\mathrm{H}}^{2}(1-f), \tag{20b}$$

which predict

$$\nu=\bar{\nu} \tag{21a}$$

$$E=\bar{E}\frac{(1-D)^{2}}{1-f} \tag{21b}$$

according to the definition of effective stress in Eq. (5).

In isotropic CDM (for both equivalent strain and equivalent elastic strain energy hypotheses), a constant Poisson's ratio is assumed (pp. 355 in Lemaitre and Chaboche, 1990). There are few experimental data to verify this conclusion although recent work on steel in Alves (1996) supports this conclusion, where the average value of elastic Poisson's ratio up to $\epsilon_{\mathrm{eq}}^{\mathrm{p}}=0.7$ is $\nu_{\mathrm{a}}=0.29$. No evidence in Alves (1996) indicates the dependence of Poisson's ratio on the damage parameter. However, results in Hansen and Schreyer (1994) showed a slight consistent decrease of Poisson's ratio with axial strain. It was noticed that Hansen and Schreyer (1994) also pointed out that the apparent Poisson's ratio remains constant with increasing damage for an isotropic model. They found that the introduction of an anisotropic model gave more encouraging results to explore the variation nature of Poisson's ratio.

Now, let us determine the thermodynamic force B defined by Eq. (8). According to Eq. (7), ψ is a function of $\epsilon_{\mathrm{ij}}^{\mathrm{e}}$, T, D and V_{k}. However, the material damage and other internal variables, V_{k}, developed during any loading/dissipative process, will retain their values during an elastic unloading process. Thus, the free energy density of a damaged material can be expressed by the elastic strain energy density of the damaged material, as shown in Eq. (9). The influence of damage on the free energy density is represented by the change of elastic parameters with the damage parameter, and the influence of plastic dissipation on the free energy density is the elastic range determined by strain hardening in a yield/loading function. Therefore, B is given by

$$B=\left.\frac{\partial(\rho\psi)}{\partial D}\right|_{\epsilon_{ij}^{\mathrm{e}}}=-\left.\frac{\partial W^{\mathrm{e}}}{\partial D}\right|_{\epsilon_{ij}^{\mathrm{e}}}. \tag{22}$$

A similar conclusion may be reached by the assumption of decoupling between plastic internal parameters and other effects, as shown by Lemaitre and Chaboche (1990, pp. 400).

By using Eq. (16a), the elastic strain energy density of damaged material is

$$W^{\mathrm{e}}=\left(\frac{W^{\mathrm{e}}}{E}\right)E, \tag{23}$$

in which

$$\frac{W^{\mathrm{e}}}{E}=g(\epsilon_{ij}^{\mathrm{e}},\nu)=\frac{9\epsilon_{\mathrm{e}}^{2}}{8(1+\nu)^{2}}\left[\frac{2}{3}(1+\nu)+3(1-2\nu)\left[\frac{2(1+\nu)\epsilon_{\mathrm{H}}^{\mathrm{e}}}{3(1-2\nu)\epsilon_{\mathrm{e}}}\right]^{2}\right]=g(\epsilon_{ij}^{\mathrm{e}},\bar{\nu}) \tag{24}$$

which is independent of the damage parameter, D, for a given elastic strain. Thus, Eqs. (22)–(24) give

$$B = -\frac{\partial W^{\mathrm{e}}}{\partial D}\bigg|_{\epsilon_{ij}^{\mathrm{e}}} = -\frac{W^{\mathrm{e}}}{E}\left(\frac{\mathrm{d}E}{\mathrm{d}D}\right) \tag{25}$$

in which, E is a function of damage parameter D that has been confirmed by many test results.

Up to now, we have not used any equivalent hypothesis to determine the value of the damaged Young's modulus E. According to the equivalent strain and equivalent elastic strain energy hypotheses, the values of E are

$$E = \bar{E}(1 - D) \tag{26a}$$

$$E = \bar{E}(1 - D)^2, \tag{26b}$$

respectively (Hansen and Schreyer, 1994). While, E is determined by Eq. (21b) when the method in the present paper is used. Thus, the values of B in Eq. (25) for these three different hypotheses are

$$B = W^{\mathrm{e}}G(D) \tag{27}$$

which, together with Eq. (25), predicts $G(D) = -\mathrm{d}(\ln E)/\mathrm{d}D$, or,

$$G(D) = \frac{1}{1 - D} \tag{28a}$$

for equivalent strain hypothesis

$$G(D) = \frac{2}{1 - D} \tag{28b}$$

for equivalent elastic strain energy hypothesis and

$$G(D) = \frac{4 - \alpha D^{\frac{3}{2}} - 3\alpha D^{\frac{1}{2}}}{2(1 - D)(1 - \alpha D^{\frac{3}{2}})} \tag{28c}$$

for the present theory, according to Eqs. (26a) and (26b) and Eq. (21b). Fig. 1 contains a comparison between the three different hypotheses and shows that the current results lay between the equivalent strain and equivalent elastic strain energy hypotheses for a valid range of the damage parameter, D.

Now consider the two terms in Eq. (11). The total mechanical dissipation energy density of a damaged material consists of two parts according to Eq. (11), i.e.

$$\phi_1 = W^{\mathrm{p}} + W^{\mathrm{d}} \tag{29}$$

where

$$W^{\mathrm{p}} = \int_0^{\epsilon_{ij}^{\mathrm{p}}} \sigma_{ij}\epsilon_{ij}^{\mathrm{p}} = \int_0^{\epsilon_{\mathrm{e}}^{\mathrm{p}}} \sigma_{\mathrm{e}}\, \mathrm{d}\epsilon_{\mathrm{e}}^{\mathrm{p}} \tag{30}$$

is the plastic strain energy density where the equivalent plastic strain is defined by Eq. (15b) when plastic strain components are used instead of elastic strain components, and the damage dissipation density is

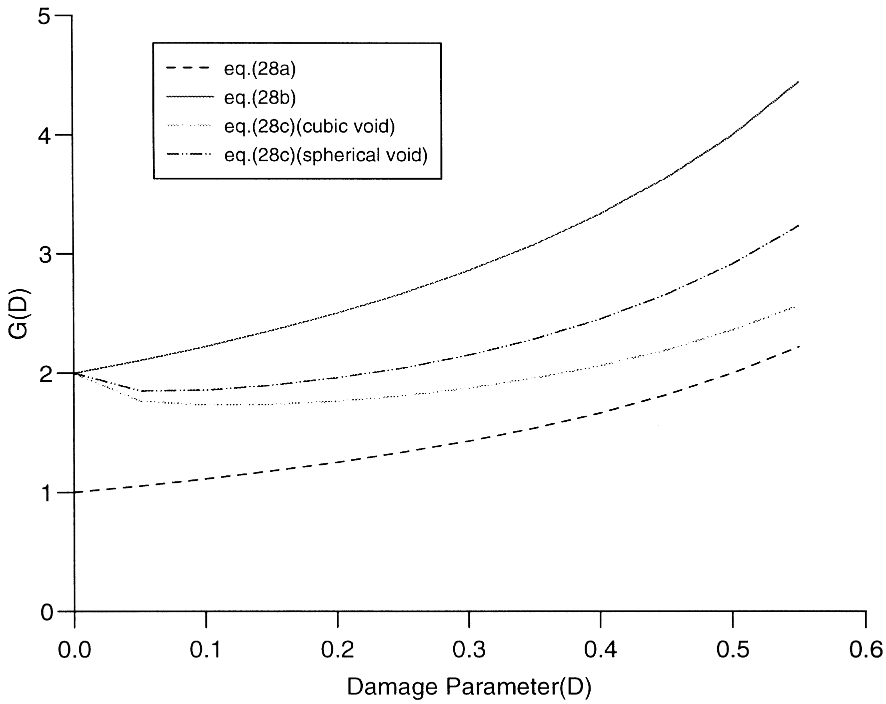

Fig. 1. Variation of the parameter G in Eq. (27) with damage parameter (D), where $\alpha=1$ for a cubic void and $\alpha=0.752$ for a spherical void, respectively.

$$W^{\mathrm{d}}=\int_0^D B\,\mathrm{d}D=-\int_0^D\left(\frac{W^{\mathrm{e}}}{E}\right)\frac{\mathrm{d}E}{\mathrm{d}D}\,\mathrm{d}D=-\frac{W^{\mathrm{e}}}{E}\int_0^D\frac{\mathrm{d}E}{\mathrm{d}D}\,\mathrm{d}D=W^{\mathrm{e}}\left(\frac{\bar{E}}{E}-1\right), \tag{31}$$

because W^{e}/E is independent of D for a given elastic strain.

The hydrostatic stress, σ_{H}, is not very significant during the ductile deformation in a tension state before a fully developed necking stage starts. In this case, the elastic strain energy density in Eq. (17) is much smaller than the plastic strain energy density, which implies that $W^{\mathrm{d}}\ll W^{\mathrm{p}}$ according to Eq. (31). Thus, the Energy Equivalent Principle II, expressed by Eq. (14) gives

$$\sigma_{ij}\,\mathrm{d}\epsilon_{ij}^{\mathrm{p}}=\bar{\sigma}_{ij}\,\mathrm{d}\bar{\epsilon}_{ij}^{\mathrm{p}}(1-f) \tag{32}$$

which is a fundamental equation in VGM and will be discussed further in section 4.

4. Applications

4.1. Plastic yield condition

The plastic yield condition defines a sub-space in stress space, within which the material behaves elastically. A general isotropic yield condition may be expressed by

$$f(\sigma_{ij})=h(V_{\mathrm{k}}), \tag{33}$$

$f(\sigma_{ij}) = h(V_k)|_{V_k=0}$ defines the initial yield surface. V_k are plastic internal variables, such as the isotropic hardening parameter. It is evident that if Eq. (33) represents a yield condition, then

$$F(f(\sigma_{ij})) = F(h(V_k)) \tag{34}$$

is also a yield condition, equivalently, if and only if F is a continuous one to one relationship for all possible values of $f(\sigma_{ij})$ and $h(V_k)$.

It is well-known that the undamaged material plasticity is initiated by the stored elastic distortional strain energy. Thus, an elastic distortional strain energy density of the damage-free matrix material may be used to define a yield condition, i.e.

$$\bar{f}(\bar{\sigma}_{ij}) = \bar{W}_d^e = \bar{h}(\bar{V}_k), \tag{35}$$

or equivalently, according to Eqs. (34) and (18a)

$$\bar{\sigma}_e = \sqrt{\frac{3\bar{E}}{(1+\bar{\nu})}\bar{h}(\bar{V}_k)} = R_0 + R(\bar{\epsilon}_e^p). \tag{36}$$

in which, R_0 is the initial yield stress of the matrix material and $R(\bar{\epsilon}_e^p)$ is the hardening function. The effective stress satisfies Eq. (36) and may be replaced by the corresponding macroscopic stress defined in Eq. (5), thus the macroscopic yield condition for a damaged material is

$$f(\sigma_{ij}, D) = \frac{\sigma_e}{1-D} = R_0 + R(\bar{\epsilon}_e^p) \tag{37}$$

which has been employed by most CDM models (Lemaitre and Choboche, 1990; Ju, 1989; Voyiadjis and Kattan, 1990; Chow and Wang, 1987b). The macroscopic yield condition expressed in Eq. (37) is not a simple replacement of $\bar{\sigma}_e$ by $\sigma_e/(1-D)$, but satisfies Eq. (19a), or the Energy Equivalent Principle I. This viewpoint was not shown in previous studies. It should be noted that $\bar{\epsilon}_e^p$ cannot be replaced by ϵ_e^p in Eq. (37) because the Energy Equivalent Principle I and the equivalent strain hypothesis are controversial, as discussed in section 4.3. The relationship between $\bar{\epsilon}_e^p$ and ϵ_e^p will be discussed in section 4.2.

Generally speaking, the definition of the effective stress in Eq. (5) only considers the geometry influence of damage on the effective resisting area. Any interactions between the damage are not included. A general definition of the effective stress is proposed here to include these influencing factors within the valid range of the CDM, i.e.

$$\bar{\sigma}_{ij} = \frac{\sigma_{ij}}{M(D)}. \tag{38}$$

in which $M(D)$ is defined as an effective resistance area factor. All previous results in the present paper may be generalized by substituting $(1-D)$ by $M(D)$. In order to obtain the common VGM yield function from the concept of effective stress in CDM, $M(D)$ is defined as

$$M(D) = \sqrt{(1 + q_3 f^{*2}) - 2q_1 f^* \cosh\left(\frac{3q_2}{2}\frac{\sigma_H}{\sigma_M}\right)} \tag{39}$$

where $q_1 = 1.5$, $q_2 = 1$ and $q_3 = q_1^2$; f^* is the modified damage volume friction, given by

$$f^* = f \quad \text{when} \quad f \le f_c$$

$$f^* = f_c + \frac{f_u - f_c}{f_F - f_c}(f - f_c) \quad \text{when} \quad f > f_c, \tag{40}$$

where $f_c = 0.15$, $f_F = 0.25$ and $f_u = 1/q_1$ have been used in several applications (Tvergaard, 1981, 1990; Tvergaard and Needleman, 1984); σ_M is the von-Mises equivalent stress for the damage-free matrix material, i.e.

$$\sigma_M = \bar{\sigma}_e = R_0 + R(\bar{\varepsilon}_e^p). \tag{41}$$

Thus, Eqs. (36) and (34) together with Eq. (38) predict a yield function

$$\Phi(\sigma_{ij}, \sigma_M, f) = \frac{\sigma_e^2}{\sigma_M^2} + 2q_1 f^* \cosh\left(\frac{3q_2}{2}\frac{\sigma_H}{\sigma_M}\right) - (1 + q_3 f^{*2}) = 0, \tag{42}$$

which is the well-known modified Gurson's model (Gurson, 1977; Tvergaard, 1981, 1990; Tvergaard and Needleman, 1984) for a porous material with spherical voids. It appears that $M(D)$ depends on both the damage development stage and the current macroscopic hydrostatic stress. In a uniaxial tensile test, $\sigma_H = \sigma_1/3$, $\sigma_M = \sigma_1/M(D)$. Thus, $\sigma_H/\sigma_M = M(D)/3$, and therefore, Eq. (39) becomes

$$M(D) = \sqrt{(1 + q_3 f^{*2}) - 2q_1 f^* \cosh\left(\frac{q_2 M(D)}{2}\right)}, \tag{43}$$

which is solved numerically. Fig. 2 gives the variation of $M(D)$ with D for the parameters given before, which is compared with $M(D) = 1-D$ used in CDM. It transpires that $M(D) = 1-D$ is close to $M(D)$

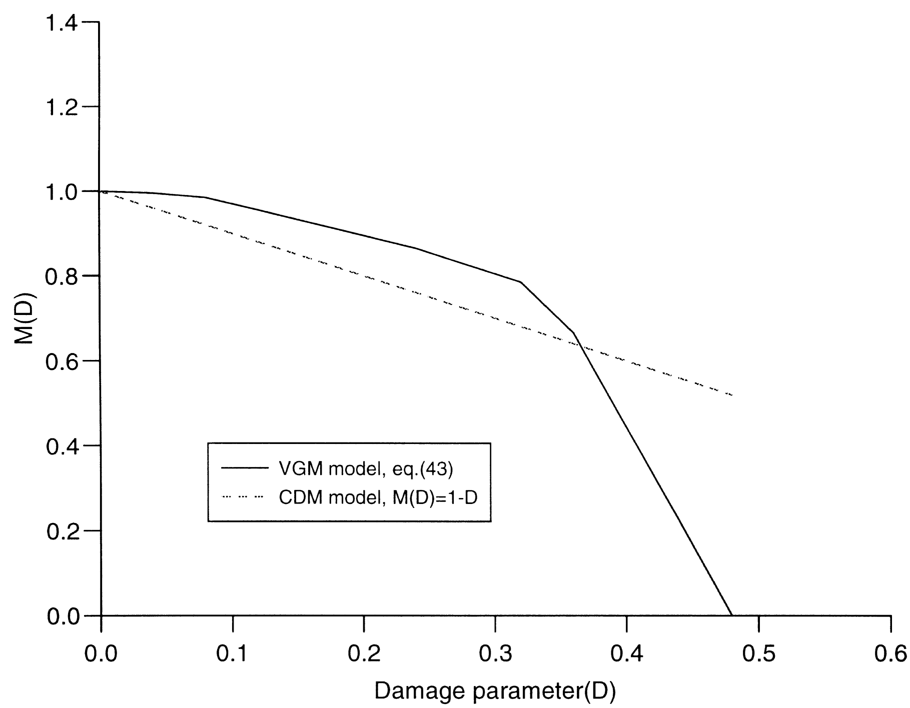

Fig. 2. Variation of the effective resistance area factor, $M(D)$, with the damage parameter, D, for VGM and CDM models.

given by Eq. (43) up to $D = 0.37$ for this particular VGM model. One important feature in Fig. 2 is that $M(D)=0$ at critical damage parameter, $f=f_F$(or $D=D_f=\alpha f_F^{3/2}$) for the modified Gurson's model, which means that the material stiffness matrix degenerates to zero when material damage approaches its critical value. This advantage allows the modified Gurson's model to be used for the whole damage development process continuously up to complete material failure.

In CDM, the yield surface includes the damage softening effect through the factor $(1-D)$. The critical values of $M(D)=(1-D)$ range from 0.15 to 0.83 for various materials according to Lemaitre and Chaboche (1990, pp. 364), which defines material failure before the stiffness matrix degenerates to zero.

The above discussion indicates that a VGM can be obtained from the foundation of CDM. However, a VGM gives a more realistic expression for the yield surface because it is obtained from an average analysis of void and aggregate arrangements, including their interactions through Eqs. (38) and (39), which are neglected in CDM models.

4.2. Macroscopic plastic flow rule

It has been shown that the plastic flow rule for damage-free matrix material is

$$d\bar{\epsilon}_{ij}^{p} = d\bar{\lambda}\frac{\partial\bar{f}(\bar{\sigma}_{ij})}{\partial\bar{\sigma}_{ij}}. \tag{44}$$

Although the macroscopic plastic flow rule of a damaged material may be obtained in other ways without using effective stress concept (Li, 1999b), we will use the concept of effective stress in the following discussion to obtain the macroscopic flow rule.

According to Eqs. (32) and (44)

$$d\epsilon_{ij}^{p} = (1-f)\frac{\bar{\sigma}_{ij}}{\sigma_{ij}}\,d\bar{\epsilon}_{ij}^{p} = \frac{1-f}{M(D)}\,d\bar{\lambda}\frac{\partial\bar{f}(\bar{\sigma}_{ij})}{\partial\bar{\sigma}_{ij}} = d\lambda\frac{\partial f(\sigma_{ij}, D)}{\partial\sigma_{ij}}, \tag{45}$$

when material damage and plasticity are decoupled, and where $d\lambda=(1-f)d\bar{\lambda}$ and $f(\sigma_{ij}, D)=\bar{f}(\bar{\sigma}_{ij})$. Eq. (45) is a normality expression of the macroscopic flow rule.

When the von-Mises yield condition in Eq. (36) is used for damage-free matrix material, following relationships are obtained from Eqs. (44) and (45)

$$d\bar{\lambda} = d\bar{\epsilon}_{e}^{p} = \sqrt{\frac{2}{3}\,d\bar{\epsilon}_{ij}^{p}\,d\bar{\epsilon}_{ij}^{p}} \tag{46a}$$

$$d\lambda = M(D)\,d\epsilon_{e}^{p} = M(D)\sqrt{\frac{2}{3}\,d\epsilon_{ij}^{p}\,d\epsilon_{ij}^{p}}, \tag{46b}$$

respectively, which have been obtained in CDM previously when $M(D)=(1-D)$. Thus, a relationship between $d\epsilon_{e}^{p}$ and $d\bar{\epsilon}_{e}^{p}$ is

$$d\epsilon_{e}^{p} = \frac{1-f}{M(D)}\,d\bar{\epsilon}_{e}^{p}. \tag{47}$$

The same result can be obtained by using Eq. (32) directly when Eq. (32) is expressed as

$$\sigma_{e}\,d\epsilon_{e}^{p} = \sigma_{ij}\,d\epsilon_{ij}^{p} = \bar{\sigma}_{e}\,d\bar{\epsilon}_{e}^{p}(1-f) \tag{48}$$

and $\bar{\sigma}_e$ is substituted by $\sigma_e/M(D)$ using Eq. (38). Eq. (48) is also a fundamental equation in VGM when $\bar{\sigma}_e$ and $d\bar{\epsilon}_e^p$ are substituted by σ_M and $d\epsilon_M$ to represent the equivalent stress and the incremental equivalent plastic strain in the damage-free matrix material (Tvergaard, 1981, 1990; Tvergaard and Needleman, 1984).

These results show that VGM can be derived from CDM according to the proposed theory. The normality requirement for macroscopic plastic flow can be proved by using the same procedure in CDM. Gurson (1977) has employed a totally different proof in which the macroscopic yield condition and flow rule are derived for a particular void shape and matrix composite by using an average technique and bound theory. The physical foundation used in VGM is rigid-plastic theory, which is less fundamental than the thermodynamic theory used in CDM. However, the average technique and a detailed void-matrix model in VGM is more realistic than the effective stress defined by Eq. (5). Thus, both theories have advantages in applications and have solved similar material damage problems but with quite different expressions.

4.3. Damage measurements

There are several different ways based on the equivalent hypotheses to measure the development of damage by measuring the degradation of Young's modulus. Equivalent strain and equivalent elastic strain energy hypotheses are used frequently for this purpose. However, it has been shown that there is a significant difference between the predictions of these two hypotheses. The degradation of Young's modulus predicted by these two hypotheses and the present theory are

$$\frac{E}{\bar{E}} = M(D) \tag{49a}$$

for the equivalent strain hypothesis

$$\frac{E}{\bar{E}} = M^2(D) \tag{49b}$$

for the equivalent elastic strain energy hypothesis

and

$$\frac{E}{\bar{E}} + \frac{M^2(D)}{1-f} \tag{49c}$$

for present theory

when the general definition of effective stress in Eq. (38) is used. Several interesting cases from Eq. (49) are now examined.

1. Differences between the various hypotheses: Hansen and Schreyer (1994) have examined the differences between the equivalent strain and equivalent elastic strain energy hypotheses. If the simple relationship $M(D) = 1 - D$ is used, the degradation of Young's modulus from three equivalent hypotheses are shown in Fig. 3, where the current results lay between the results from the equivalent strain and equivalent elastic strain energy hypotheses. It implies that the equivalent strain hypothesis overestimates the material stiffness, while the equivalent elastic strain energy hypothesis underestimates the material stiffness.
2. Measurement of material damage: In Eq. (49), E and $\bar{E}$ are both measurable and therefore, many people have used them to obtain the material damage parameter, D. When material damage is defined by Eqs. (1)–(3), it has a unique meaning. However, when $M(D)$ is assumed to be $(1-D)$, as

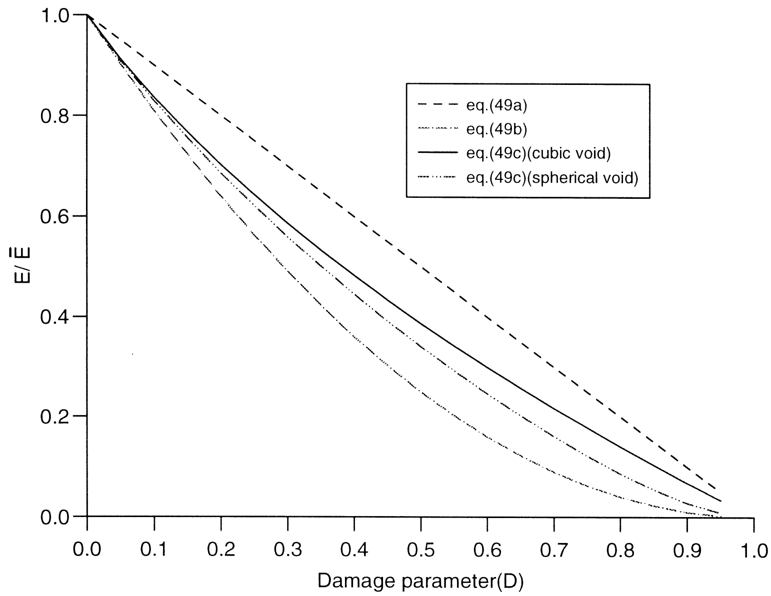

Fig. 3. Degradation of Young's modulus with damage parameter, D, for various equivalent hypotheses.

in many CDM models, there is a large difference in the damage parameter D. For example, Alves (1996) measured the damage parameters using different techniques, which revealed a significant difference between the geometry measurement and other methods, as shown in Table 1 for mild steel. Although, the author claimed that the damage parameter might be seen as an adjustable parameter, which has also been pointed out by Kachanov (1994), these inconsistencies are still a significant difficulty for CDM. However, if $M(D)$ is treated as a general function of D using Eq. (38), this difficulty can be resolved. According to the new definition of effective stress in Eq. (38), the damage parameter defined by Eq. (5) under various equivalent hypotheses is $D^{\mathrm{M}}=(1-M(D))$, which is not the damage parameter, D, defined geometrically by Eqs. (1)–(3), but a function of the damage parameter. Because the only parameter which needs to be evaluated is D^{M} throughout the use of CDM, even for defining material failure. Thus, the existing models of CDM, no matter which equivalent hypothesis is used, still gives reasonable results if D^{M} is used consistently in both theoretical model and experimental programm. However, if one would like to use the real value of the damage parameter, D, there is a significant difference among various definitions of D. For example, the difference in damage parameter, D, obtained by measuring Young's modulus and measuring the void area, is around 10^2–10^3 according to Table 1. In the modified Gurson's model,

Table 1
Static critical damage parameters for mild steel according to different definitions and experimental techniques (Alves, 1996)

Damage parameter	Values	Technique
$D_{\mathrm{E}\epsilon}$	0.45	Equivalent strain hypothesis
D_{EW}	0.26	Equivalent elastic strain energy hypothesis
D_{V}	0.20	Voltage measurement
D_{HV}	0.041	Hardness measurement
D_{S}	0.0072	Measurement of the voids area

the critical damage volume friction is $f_F = 0.25$, which corresponds to a critical damage area density $D_F = 0.48$, according to Eq. (4). This value is much larger than the value in Alves (1996), where the measurement of the voids area after material failure gives $D_s = 0.0072$. If they both depend on a geometrical definition of the damage parameter in section 2.1, such a large difference, despite the difference between materials, appears to be unacceptable. Thus, systematic experimental studies are necessary for assessing the reality of the various damage parameters in different damage models.

5. Conclusions

Energy equivalence principles are established based on an energy correlation hypothesis in the present paper. These principles are used to obtain mechanical property relationships between a damaged material and its virgin state of matrix material. The proposed theory unifies the continuum damage mechanics (CDM) and void growth model (VGM) on the same thermodynamic foundation.

The advantages and disadvantages of both CDM and VGM are discussed based on the proposed theory. Two well-known equivalence hypotheses in damage mechanics, i.e. equivalent strain and equivalent elastic strain energy hypotheses, are examined and compared with the energy equivalent principles obtained in the present paper. The predicted results based on the present theory lay between the results obtained from equivalent strain and equivalent elastic strain energy hypotheses.

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Appendix A

Consider a group of regularly distributed cubic voids with characteristic length b within a cubic

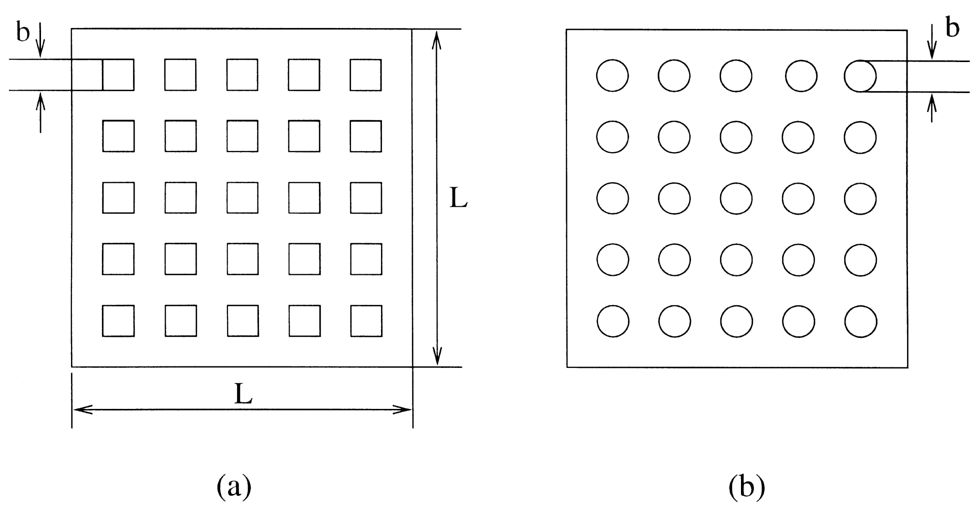

Fig. A1. A plane view of regularly distributed voids within a cube, (a) cubic void, and (b) spherical void.

block with characteristic length L, shown in Fig. A1(a), the line, area and volume densities of the damage are

$$D = \frac{n^2 b^2}{L^2}, \quad f = \frac{n^3 b^3}{L^3} \quad \text{and} \quad l = \frac{nb}{L} \tag{A1}$$

according to Eqs. (1)–(3), in which n is the void numbers within the length L.

Similarly, the line, area and volume densities of damage are

$$D = \frac{\pi n^2 b^2}{4L^2}, \quad f = \frac{\pi n^3 b^3}{6L^3} \quad \text{and} \quad l = \frac{nb}{L} \tag{A2}$$

for a group of regularly distributed spherical voids in Fig. A1(b).

Eq. (4) is obtained from Eqs. (A1) and (A2).

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